

# Bilinear gauge operators

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## Abstract

We construct a family of bilinear differential operators which satisfy certain gauge properties. These operators can be naturally associated with  $q$ -deformations of classical integrable hierarchies. In particular, we consider the case when gauge function = Hurwitz-type partition function.

## 1 Introduction

Our aim is to represent bilinear Hirota-type differential operators of the form

$$D^n = \sum_{0 \leq i, j \leq n} C_{i,j}(q) \frac{\partial^i}{\partial x^i} \frac{\partial^j}{\partial x'^j} \Big|_{x=x'}, \quad q \in \mathbb{C}, \quad (1)$$

which satisfy gauge condition for some fixed  $g$ :

$$D \langle g f_1, g f_2 \rangle = g^{2+s} D \langle f_1, f_2 \rangle \quad (2)$$

where  $f_1(x), f_2(x')$  are smooth complex-valued functions,  $s \in \mathbb{Z}$  and  $g$  has perturbative expansion in powers of  $\hbar = \log(q)$ :

$$g(x, e^\hbar) = \sum_{0 \leq i < \infty} S_i(x) \hbar^i \quad (3)$$

Let  $S_0(x) = e^x$ , so when  $\hbar = 0$ ,  $D^n$  coincides with the ordinary Hirota derivative [1] (it is uniquely defined using gauge condition). We claim that such operators can be used as building blocks for some interesting equations (relations with  $q$ -Schur functions are mostly conjectured, see [2, 5] for more reference).

Of course, it can be generalized to multi-linear multi-dimensional case (although there may be some obstacles, and we'll show this later). "Good" choice of gauge function  $g$  (a proper sum of  $q$ -exponentials) can lead to non-trivial properties. One of our goals is to understand the structure of bilinear operators of the type

$$P^m(D_{(g)}^1, D_{(g)}^2, \dots) = 0 \quad (4)$$

where  $P$  is a quasi-homogeneous polynomial (monomials are labeled by partitions of  $m$ ) with  $q$ -deformed coefficients. We add  $(g)$  as lower index to emphasise dependence on a gauge.

## 2 $q$ -exponential case

Before producing the general construction, consider the following example. Choose  $g(x, q)$  as the  $q$ -exponential:

$$e_q^x = \sum_{k=0}^{\infty} \frac{(1-q)^k}{(q; q)_k} x^k, \quad (5)$$

where  $(a, q)_k$  is a  $q$ -shifted factorial

$$(a, q)_k = \prod_{s=0}^{k-1} (1 - aq^s), \quad k = 1, 2, \dots \quad (6)$$

Recall definition of  $q$ -derivative:

$$d_q(f) = \frac{f(x) - f(qx)}{(q-1)x} \quad (7)$$

Consider the following  $q$ -difference operator:

$$\Delta_q^1(f_1(x), f_2(x')) = \frac{(-f_1(x) + f_1(qx)) f_2(x')}{(q-1)x} - \frac{(-f_2(x') + f_2(qx')) f_1(x)}{(q-1)x'} \quad (8)$$

It can be checked that (8) does not satisfy gauge condition (opposite to its differential limit). Now substitute  $q = e^{\hbar}$  in (5) and expand in powers of  $\hbar$ :

$$e_q^x = e^x \left( 1 - 1/4 x^2 \hbar + (1/9 x^3 + 1/32 x^4) \hbar^2 + \left( 1/48 x^2 - 1/16 x^4 - 1/36 x^5 - \frac{1}{384} x^6 \right) \hbar^3 + O(\hbar^3) \right) \quad (9)$$

We also rewrite (1) as  $\hbar$ -series:

$$D^n = \sum_{0 \leq k < \infty} \sigma_k(\partial^0, \partial^1, \dots, \partial^{k+1}, x) \hbar^k, \quad (10)$$

where  $\sigma_i$  are (to be found) bilinear differential operators with polynomial coefficients. Consider that the following gauge property holds:

$$\sum_{k=0}^{\infty} \sigma_k e_q^x \langle f_1, f_2 \rangle \hbar^k = e^{2x} \sum_{k=0}^{\infty} \sigma_k \langle f_1, f_2 \rangle \hbar^k \quad (11)$$

It follows that  $\sigma_i$  should satisfy the functional equation:

$$\sigma_i \langle f_1, f_2 \rangle = \sigma_i e^x \langle f_1, f_2 \rangle + \sum_{j=1}^i \text{coeff}(\sigma_{i-j} e_q^x \langle f_1, f_2 \rangle, \hbar^j) \quad (12)$$

We choose initial condition:

$$\sigma_0 = D(f_1, f_2) = g_1 g_{2x} - g_2 g_{1x} \quad (13)$$

Now our coefficients can be written as

$$\sigma_i = \sum_{k=2}^{i+1} a_{i,k}(x) Q_k(\partial), \quad Q_j = g_1 g_{2jx} - g_2 g_{1jx} \quad (14)$$

so we can compare  $D^n$  with  $\Delta_q$  expansion! The first few coefficients are

$$\begin{array}{l|l}
\sigma_1 & 1/4 x^2 Q_2 \\
\sigma_2 & x^2 \left( \frac{1}{72} x Q_2 + 1/24 x^2 Q_3 \right) \\
\sigma_3 & x^3 \left( \frac{1}{288} \frac{(5x^2-6)Q_2}{x} + \frac{1}{216} x^2 Q_3 + \frac{1}{192} x^3 Q_4 \right) \\
\sigma_4 & x^4 \left( -\frac{1}{43200} \frac{(-300+503x^2)Q_2}{x} + \left( \frac{23}{3888} x^2 - \frac{1}{144} \right) Q_3 + \frac{1}{1152} x^3 Q_4 + \frac{1}{1920} x^4 Q_5 \right) \\
\sigma_5 & x^5 \left( \frac{1}{259200} \frac{(540+2431x^4-2175x^2)Q_2}{x^3} + \left( \frac{5}{2592} - \frac{173}{48600} x^2 \right) Q_3 + \left( -\frac{1}{768} x + \frac{47}{41472} x^3 \right) Q_4 + \right. \\
& \left. + \frac{1}{8640} x^4 Q_5 + \frac{1}{23040} x^5 Q_6 \right)
\end{array}$$

### 3 Gauge operators $D_{(g)}$

Again, we are dealing with gauge functions analytic in  $\hbar$ :

$$g(x_1, x_2, \dots, \hbar) = e^{x + \sum_{i=1}^{\infty} S_i(x) \hbar^i}, \quad (15)$$

We are going to construct  $D_{(g)}^n$  in a perturbative way, using the expansion

$$D_{(g)}^n = D^n + \sum_{i=1}^{\infty} \sigma_i \hbar^i, \sigma_i = \sum_{0 \leq l, m \leq i+1} a_i \frac{d^l}{dx^l}(f_1) \frac{d^m}{dx^m}(f_2) \quad (16)$$

( $D^n$  = Hirota derivative), s.t. the gauge property (2) is satisfied. For simplicity we take  $s = 0$ . Note that (2) implies

$$D_{(g)}^n \langle g, g \rangle \equiv 0, \quad \forall n \in \mathbb{Z} \quad (17)$$

Choose the first 2 coefficients as

$$D_{(g)}^0 \equiv D^0 = f_1 f_2, \quad D_{(g)}^1 \equiv D^1 = f_1' f_2 - f_1 f_2' \quad (18)$$

Then, every  $D_{(g)}^n$  is a finite bilinear differential operator, acting on a pair of functions  $f_1, f_2$ . Now we are ready to represent the main results, achieved with symbolic computations.

**Thm.1** *Let  $m$  be even, then we have a symmetric formula:*

$$D_{(g)}^m(f, 1) \equiv D_{(g)}^m(1, f) = \frac{d^m}{dx^m} f(x) + \sum_{k=0}^{\frac{1}{2}m-1} I_{\frac{1}{2}m-k}(x, \hbar) \frac{d^{2k+1}}{dx^{2k+1}} f(x), \quad (19)$$

where  $I_s(x, \hbar) = I_s(g)$  are rational in  $g, g', \dots, g^{(m)}$  (note here we equal all arbitrary constants to zero just for simplicity, thus killing all higher order  $D$ 's in the formula).

For instance,

$$D_{(g)}^2 = D^2 + I_1(g)(f_1 f_2)', \quad (20)$$

where  $I_1(g) = \sum I_{1,i}(x) \hbar^i$ . It is easy to see that generating function for  $\{I_{1,i}\}$  is a natural logarithm:

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$$\begin{array}{c}
\int I_{1,i} dx \text{ value} \\
\hline
\begin{array}{c}
-S_{1;x} \\
1/2 S_{1;x}^2 - S_{2;x} \\
-1/3 S_{1;x}^3 + S_{2;x} S_{1;x} - S_{3;x} \\
1/4 S_{1;x}^4 + S_{3;x} S_{1;x} - S_{2;x} S_{1;x}^2 - S_{4;x} + 1/2 S_{2;x}^2 \\
-1/5 S_{1;x}^5 - S_{2;x}^2 S_{1;x} + S_{4;x} S_{1;x} - S_{3;x} S_{1;x}^2 + S_{2;x} S_{1;x}^3 + S_{3;x} S_{2;x} - S_{5;x} \\
\vdots
\end{array}
\end{array}$$

$$G(z) = \log \left( \sum_{i=1}^{\infty} t_i z^i \right), S_{j;x} \leftrightarrow \frac{\sum_{i=0}^{\infty} t_i \frac{i!}{(i-j)!}}{\sum_{i=0}^{\infty} t_i} \quad (21)$$

Then,  $I_2(g)$  can be expressed in terms of  $I_1(g)!$

$I_{2,i}$ value
$I_{1,x,x}$
$I_{2,x,x} + 1/2 I_{1,x} I_1$
$I_{3,x,x} + 1/2 I_{2,x} I_1 + 1/2 I_{1,x} I_2 - \frac{5}{36} I_1^3$
$I_{4,x,x} + 1/2 I_{3,x} I_1 + 1/2 I_{1,x} I_3 + 1/2 I_2 I_{2,x} - \frac{5}{12} I_2 I_1^2$
$I_{5,x,x} + 1/2 I_{4,x} I_1 + 1/2 I_{1,x} I_4 + 1/2 I_2 I_{3,x} + 1/2 I_3 I_{2,x} - \frac{5}{12} I_3 I_1^2 + \frac{5}{72} I_2^2 I_1$
$\vdots$

Summarizing these facts, we can write the first few  $I$ 's in closed form:

$I_1(g) = -\frac{g_{x,x}}{g_x} + \frac{g_x}{g}$
$I_2(g) = I_{1,x,x} + \frac{3}{2} I_{1,x}^2 - 5 I_1^3$
$I_3(g) = I_{1,x,x,x} + 5 I_{1,x} I_{1,x,x} + 10 I_1 I_{1,x,x,x} + 15 I_1 I_{1,x}^2 - 65 I_1^2 I_{1,x,x} - 250 I_1^3 I_{1,x} + 271 I_1^5$

Note that

$$f_1''' f_2 - (f_1' f_2')' + f_1 f_2''' = \frac{d}{dx} D^2(f_1, f_2) \quad (22)$$

Solution for  $m \leq 8$  in explicit form:

$$D_q^4 = D^4 + I_2(g)(f_1 f_2)' - \frac{1}{6} I_1(g)(f_1''' f_2 - (f_1' f_2')' + f_1 f_2''') + R_4(g)(f_1' f_2'), \quad (23)$$

$$D_q^6 = D^6 + I_3(g)(f_1 f_2)' + \frac{1}{15} I_2(g)(f_1''' f_2 - (f_1' f_2')' + f_1 f_2''') + \frac{1}{15} I_1(g)(D^4(f_1, f_2))' + \quad (24)$$

$$+ R_6(g)(f_1' f_2') + R_4(g)(-c_1(f_1'' f_2'') + c_2(f_1' f_2''' + f_1''' f_2')),$$

$$D_q^8 = D^8 + \left[ I_4(g) \frac{d}{dx} D^0 + \frac{1}{28} I_3(g) \frac{d}{dx} D^2 + \frac{1}{70} I_2(g) \frac{d}{dx} D^4 + \frac{1}{28} I_1(g) \frac{d}{dx} D^6 \right] (f_1, f_2) + R_8 D^0(f_1', f_2') \quad (25)$$

where  $R_4(g) = h^2(-24 S_{1;x,x} (S_{1;x} S_{1;x,x} - S_{2;x,x})) + h^3(12 S_{2;x,x}^2 + 36 S_{1;x,x}^2 S_{1;x}^2 - 24 S_{1;x,x}^2 S_{2;x} + 24 S_{1;x,x} S_{3;x,x} - 48 S_{1;x,x} S_{2;x,x} S_{1;x}) + O(\hbar^4)$ , and  $R_6(g) = c_1(I_1 I_{1,x,x})_h + c_2(I_1^2 I_{1,x})_{h,h}$ ,  $c_i$  - complex constants.

Remarkable fact is that  $R_4(g)$  satisfy the following equation:

$$\sum_{i=1}^{\infty} \frac{\partial}{\partial \lambda_i} R_4(g, \lambda_1, \lambda_2, \dots) = I_1(g) \frac{\partial}{\partial h} I_1(g), \quad \lambda_0 = 1, \lambda_i = \lambda_{i-1} - \frac{1}{\text{binomial}(i+2, 2)}, \quad (26)$$

We conjecture that similar equalities should hold for any  $R_j(g)$ .

Now we are ready to represent the general formula.

**Thm.2** Gauge operators  $D_{(g)}$  are expressed as follows:

$D_{(g)}^{2k} = D^{2k} + \sum_{1 \leq i < k} c_{k-i} I_{k-i}(g) \frac{d}{dx} (D^{2i}(f_1, f_2)) + R_{2k}(g) D^0(f_1', f_2')$	(27)
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$D_{(g)}^{2k+1} = D^{2k+1} + \sum_{1 \leq i < k} c_{k-i} I_{k-i}(g) (D^{2i+1}(f_1', f_2')) + R_{2k+1}(g) D^0(f_1', f_2')$	(28)
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Note here that for odd powers  $D_{(g)}^{2k+1} \langle f_1, f_2 \rangle \neq D_{(g)}^{2k+1} \langle f_2, f_1 \rangle$ , which is not true for ordinary  $D$ 's.

## 4 Speculations

Let  $g$  be a series of certain combinatorial nature. As an example, we consider Hurwitz-Kontsevich genertaring function [3, 4]:

$$g = e^{\hbar \hat{W}_0} e^{p_1} \quad (29)$$

where  $\hat{W}_0$  is the cut-and-join operator:

$$\hat{W}_0 = 1/2 \sum_{b=1}^{\infty} \left( \sum_{a=1}^{\infty} (a+b) p_a p_b D p_{a+b} + a b p_{a+b} D p_a D p_b \right), \quad D p_i = \frac{\partial}{\partial p_i} \quad (30)$$

Now take  $\log g = H(\bar{p}, q)$ , where  $H(\bar{p}, q)$  – Hurwitz partition function. Its first few terms are:

$$\begin{aligned} g_{HK} = & \left( 1 + 1/2 p_2 \hbar + 1/2 \left( 1/4 p_2^2 + p_3 + 1/2 p_1^2 \right) \hbar^2 + 1/6 \left( 1/2 p_2 + 4 p_4 + 3/4 p_2 p_1^2 + 3/2 p_2 p_3 + 1/8 p_2^3 + 4 p_2 p_1 \right) \hbar^3 + \right. \\ & + 3 p_3^2 + 1/16 p_2^4 \left. \right) \hbar^4 + \frac{1}{120} \left( 5/4 p_3 p_2^3 + 40 p_2 p_1 + 15/2 p_3 p_1^2 p_2 + 40 p_3 p_4 + \frac{125}{2} p_5 p_2 + 10 p_2^3 p_1 + \frac{125}{4} p_2^3 + 20 p_1^2 p_4 + \right. \\ & \left. + 1/32 p_2^5 + \frac{495}{4} p_2 p_1^2 + 160 p_4 + 1/2 p_2 + 10 p_4 p_2^2 + 5/8 p_2^3 p_1^2 + 15/2 p_3^2 p_2 \right) \hbar^5 + O(\hbar^6) \left. \right) e^{p_1} \end{aligned}$$

Then we have:

$$H(\bar{p}, q), \quad D_{(g)}^2 = D^2(f_1, f_2) - \frac{H_{2p_1}}{H_{p_1}} \left( (f_1)_{p_i} f_2 + \delta \cdot f_1(f_2)_{p_j} \right), \quad (31)$$

where  $\delta = 0$  or  $1$ , and the coefficient

$$I_1(g) = -\frac{H_{2p_1}}{H_{p_1}} = \frac{1}{2} \hbar^2 + \left( -\frac{1}{24} - \frac{3}{4} p_1 \right) \hbar^4 - \frac{5}{3} p_2 \hbar^5 + \dots \quad (32)$$

In these formulas  $I_k(g)$  are expressed in terms of logarithmic derivative of Hurwitz-Kontsevich function and its derivatives. Now we can check that for the following commutatuor

$$[D_{(g)}^m(f_1, 1), D_{(g)}^n(f_1, 1)], \quad m, n \text{ even} \quad (33)$$

all coefficients also depend on  $\partial \log H$  in a non-trivial way. We conjecture that, using this formalism, one can rediscover identities for Hurwitz function. Also note that if  $g$  is a non-trivial tau-function itself, then  $D_{(g)}^m(g, g)$  degenerates to some KP-like equation, while  $D_{(g)}^m(f_1, f_2)$  corresponds to its minor deformation.

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